Deviations from Off-Diagonal Long-Range Order and Mesoscopic Condensation in One-Dimensional Quantum Systems

SISSA

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11/05/2018

Outline

- Off-Diagonal Long-Range Order (ODLRO)
- ODLRO in 1D: Introduction to the Lieb-Liniger Model
- One-body density matrix for Lieb-Liniger bosons
- One-body density matrix for Lieb-Liniger anyons (including 1D hard-core anyons)
- Conclusions

Outline

- Off-Diagonal Long-Range Order (ODLRO)

  O. Penrose, L. Onsager, *Phys. Rev., 104 (1956) 576*

  L. P. Pitaevskii, S. Stringari, *Bose-Einstein condensation and Superfluidity,* Oxford University Press
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Density Matrix

One-Body Density Matrix (OBDM):

$$H\chi_N(\overrightarrow{r_1},\overrightarrow{r_2},\ldots,\overrightarrow{r_N}) = E\chi_N(\overrightarrow{r_1},\overrightarrow{r_2},\ldots,\overrightarrow{r_N})$$

$$\rho(\vec{r},\vec{r}') \equiv N\int d\overrightarrow{r_2}\ldots d\overrightarrow{r_N}\,\chi_N^*(\vec{r},\overrightarrow{r_2},\ldots,\overrightarrow{r_N})\,\chi_N(\vec{r}',\overrightarrow{r_2},\ldots,\overrightarrow{r_N})$$

Diagonal OBDM:

$$\rho(\vec{r}) \equiv \rho(\vec{r},\vec{r})$$

Normalization $\int\rho(\vec{r})\,d\vec{r} = N$

Off-Diagonal Long-Range Order

$$\int \rho(\vec{r},\vec{r}')\, \varphi_j(\vec{r}')\, d\vec{r}' = \lambda_j\, \varphi_j(\vec{r}), \qquad \lambda_j \in \mathbb{R}, \qquad \sum_j \lambda_j = N, \qquad \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots$$

natural orbitals (φ_j) — occupation numbers (λ_j)

$$\int \varphi_j(\vec{r})\, \varphi_k(\vec{r})\, d\vec{r} = \delta_{j,k}, \quad \sum_j \varphi_j(\vec{r})\, \varphi_j(\vec{r}') = \delta(\vec{r}-\vec{r}')$$

$$\rho(\vec{r},\vec{r}') = \sum_i \lambda_i \varphi_i^*(\vec{r})\varphi_i(\vec{r}')$$

$\lambda_0 \propto N$ — BEC — ODLRO

$\lambda_0 \propto N^0$ — Fermions

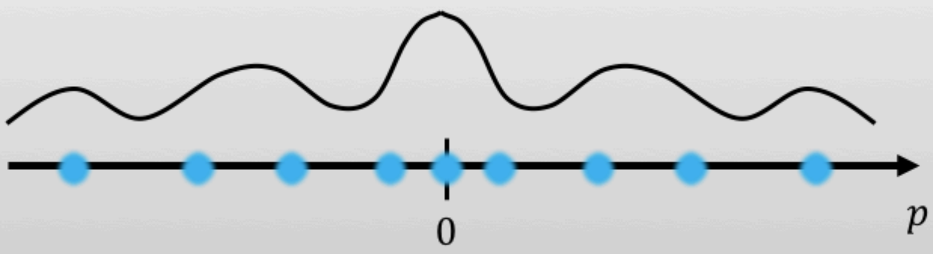
$\lambda_0 \propto N^{\mathcal{C}}$ $\quad 0 \leq \mathcal{C} \leq 1$ — Deviations from ODLRO

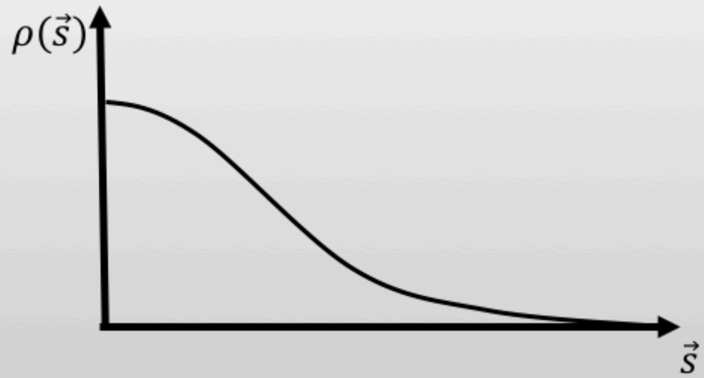
Momentum distribution

$$n(\vec{p}) = \frac{1}{(2\pi\hbar)^3}\int d\vec{r}'\int \rho(\vec{r},\vec{r}')\, e^{i\vec{p}\cdot(\vec{r}'-\vec{r})/\hbar}\, d\vec{r}$$

Normalization $\int n(\vec{p})\, d\vec{p} = N$

Homogeneous System $\rho(\vec{s}=\vec{r}-\vec{r}') = \frac{1}{V}\int n(\vec{p})\, e^{i\vec{p}\cdot\vec{s}/\hbar}\, d\vec{p} \xrightarrow{|\vec{s}|\to\infty} 0$ NO ODLRO



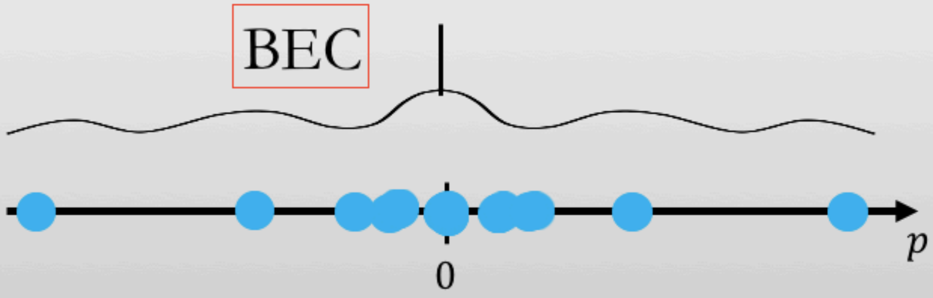


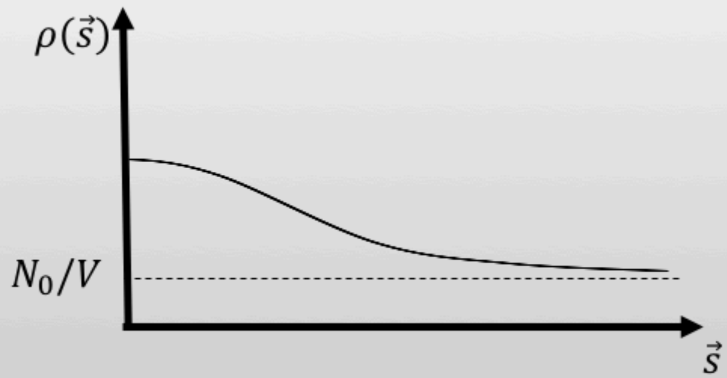
Momentum distribution

$$n(\vec{p}) = \frac{1}{(2\pi\hbar)^3} \int d\vec{r}\,' \int \rho(\vec{r}, \vec{r}\,')\, e^{i\vec{p}\cdot(\vec{r}\,' - \vec{r})/\hbar}\, d\vec{r}$$

Normalization $\int n(\vec{p})\, d\vec{p} = N$

Homogeneous System $\rho(\vec{s} = \vec{r} - \vec{r}\,') = \frac{1}{V} \int n(\vec{p})\, e^{i\vec{p}\cdot\vec{s}/\hbar}\, d\vec{p} \xrightarrow{|\vec{s}| \to \infty} \frac{\lambda_0}{V}$ ODLRO





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- ODLRO in 1D: Introduction to the Lieb-Liniger Model

  E.H. Lieb, W. Liniger, *Phys. Rev., 130 (1963) 1605*

  C.N. Yang, C.P. Yang, *J. Math. Phys., 10 (1969) 1115*
- One-body density matrix for Lieb-Liniger bosons
- One-body density matrix for Lieb-Liniger anyons (including 1D hard-core anyons)
- Conclusions

Lieb-Liniger Model

$\hbar = 2m = 1$

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} + 2c \sum_{1 \le i < j \le N} \delta(z_i - z_j)$$

$$\boxed{\gamma \equiv \frac{c}{n_{1D}}}$$

Dimensionless coupling

Low density $\Rightarrow$ Tonks-Girardeau Gas ($\gamma \to \infty$), $\chi_N \big|_{z_i = z_j} = 0$ M. Girardeau, *J. Math. Phys., 1 (1960) 516*

PBC: $\chi_N(z_1 + L, \dots, z_N) = \chi_N(z_1, \dots, z_N)$

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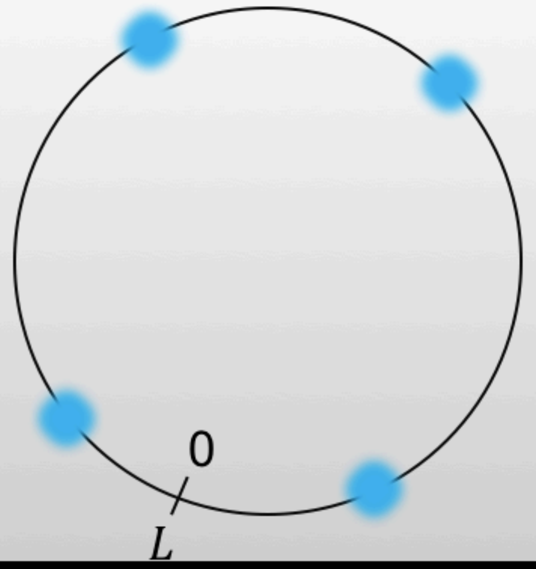

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$$\lambda_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right) + \frac{2}{L}\sum_{k=1}^{N} \arctan\left(\frac{\lambda_j - \lambda_k}{c}\right), \qquad j = 1, \dots, N$$

$$\chi_N(z_1, \dots, z_N) = \mathcal{N} \det\left(e^{i\lambda_j z_m}\right) \prod_{n<l} [\lambda_l - \lambda_n - ic\, sign(z_l - z_n)]$$

Bethe ansatz solution

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Predictions for $\mathcal{C}$

$$\frac{n(p=0)}{L} \propto N^{\mathcal{C}}$$

$$n(p) = \frac{L}{2\pi}\int_0^L \rho(z)\, e^{ipz}\, dz$$

Large distance $\rho(z \gg 1)$ ⟹ Small momenta $n(p \approx 0)$

Luttinger Liquid prediction: $\rho(z \to \infty) \propto z^{-1/2K}$

$$K(\gamma) = \frac{v_F}{s(\gamma)}$$

$$p_{min} \approx \frac{2\pi}{L} \propto N^{-1} \qquad n_{1D} = \frac{N}{L}$$

$$\frac{n(p \to 0)}{L} \propto \frac{1}{p^{1-1/2K}} \propto N^{1-1/2K}$$

F.D.M. Haldane, *Phys. Rev. Lett., 47 (1981) 1840*
T. Giamarchi, *Quantum Physics in One Dimension*

Predictions for $\mathcal{C}(\kappa)$

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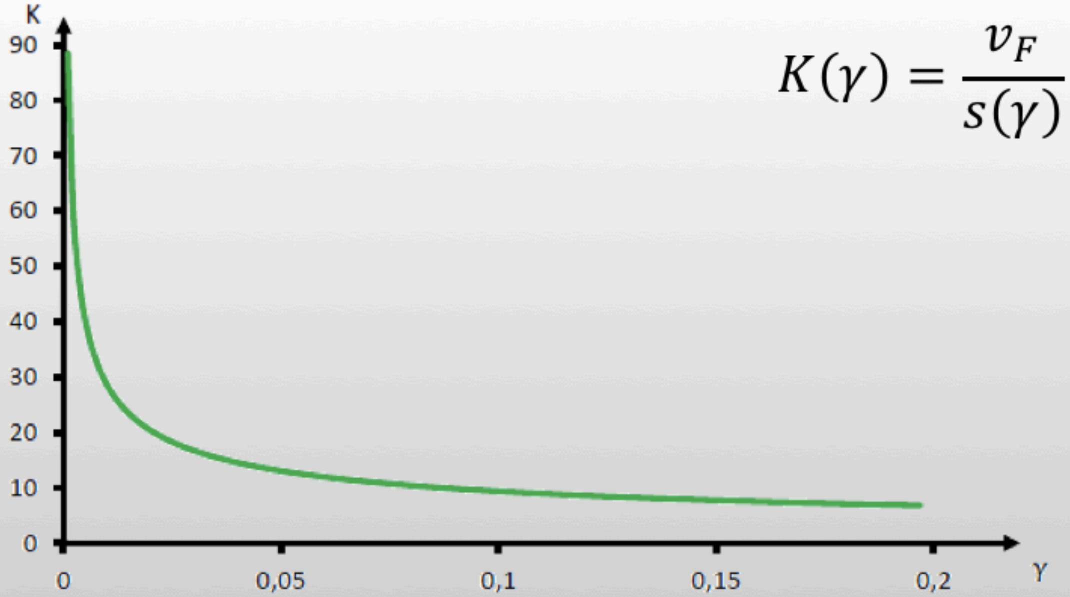
$$n(p) = \frac{L}{2\pi}\int_0^L \rho(z)\, e^{ipz}\, dz$$

Large distance $\rho(z \gg 1)$ ⟹ Small momenta $n(p \approx 0)$

$$\mathcal{C}(\gamma) = 1 - \frac{1}{2K}$$

Deviation from ODLRO



$$K(\gamma) = \frac{v_F}{s(\gamma)}$$

F.D.M. Haldane, *Phys. Rev. Lett., 47 (1981) 1840*
T. Giamarchi, *Quantum Physics in One Dimension*

Lieb-Liniger Bosons

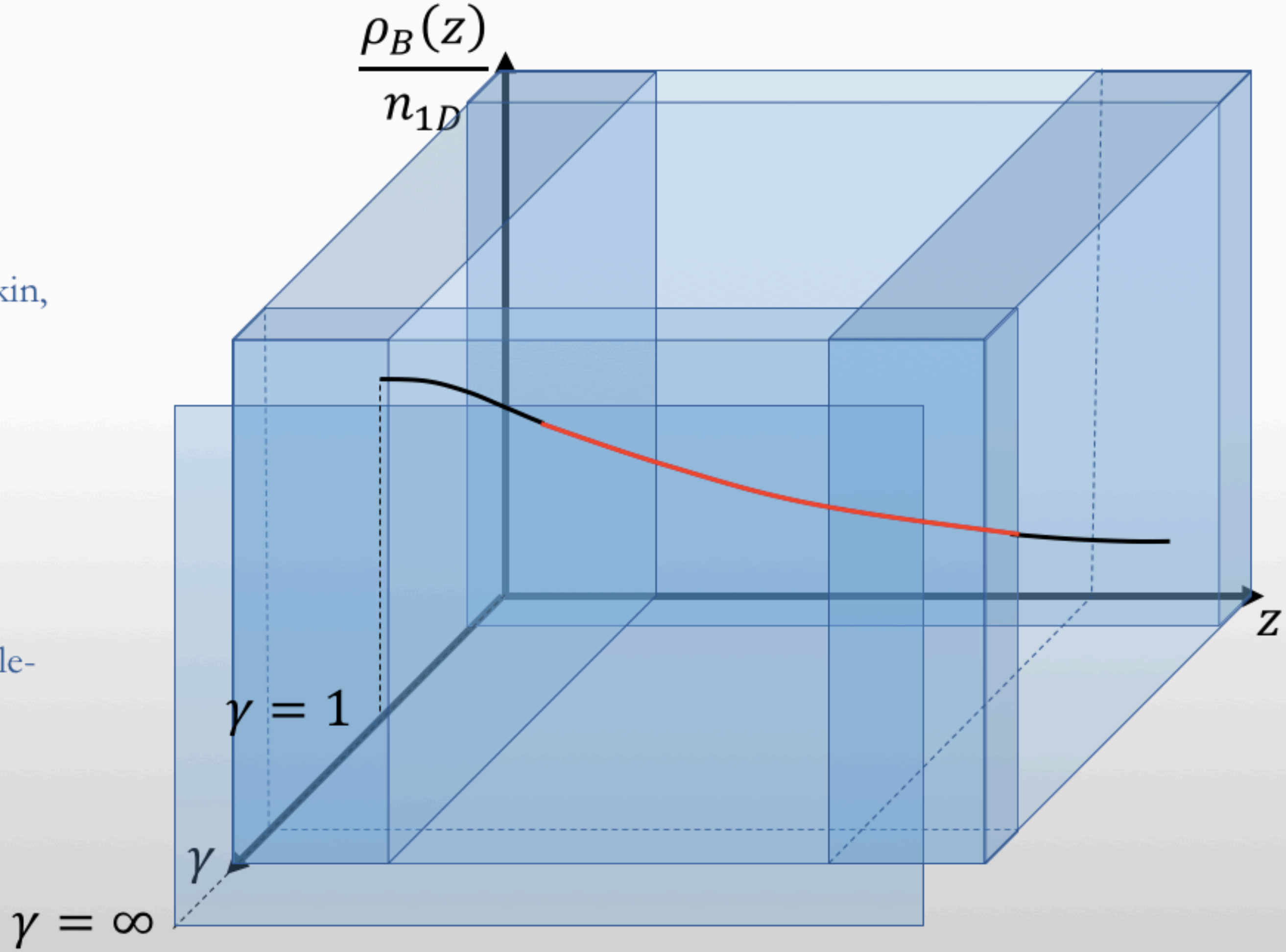
Small γ: C. Mora, Y. Castin,
Phys. Rev. A, 67 (2003) 053615.

Large γ: M. Jimbo, T. Miwa,
Phys. Rev. D, 24 (1981) 3169.
P.J. Forrester, N.E. Frankel, M.I. Makin,
Phys. Rev. A, 74 (2006) 043614.

TG limit: A. Lenard,
J. Math. Phys., 5 (1964) 930.

Small z: M. Olshanii, V. Dunjko,
Phys. Rev. Lett., 91 (2003) 090401.

Large z: A. Shashi, M. Panfil, J.-S. Caux, I. Alexander,
Phys. Rev. B, 85 (2012) 155136.

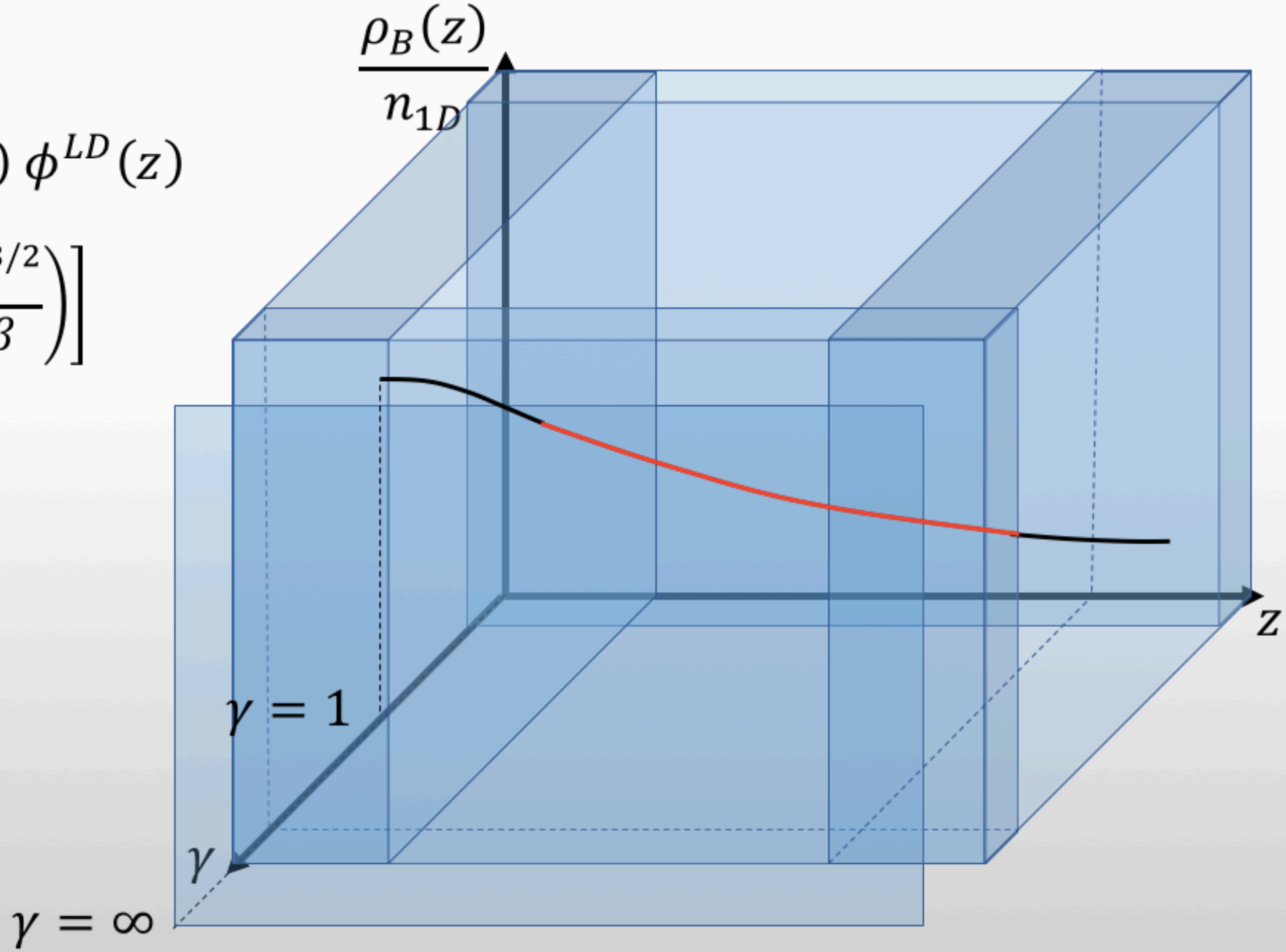


Lieb-Liniger Bosons

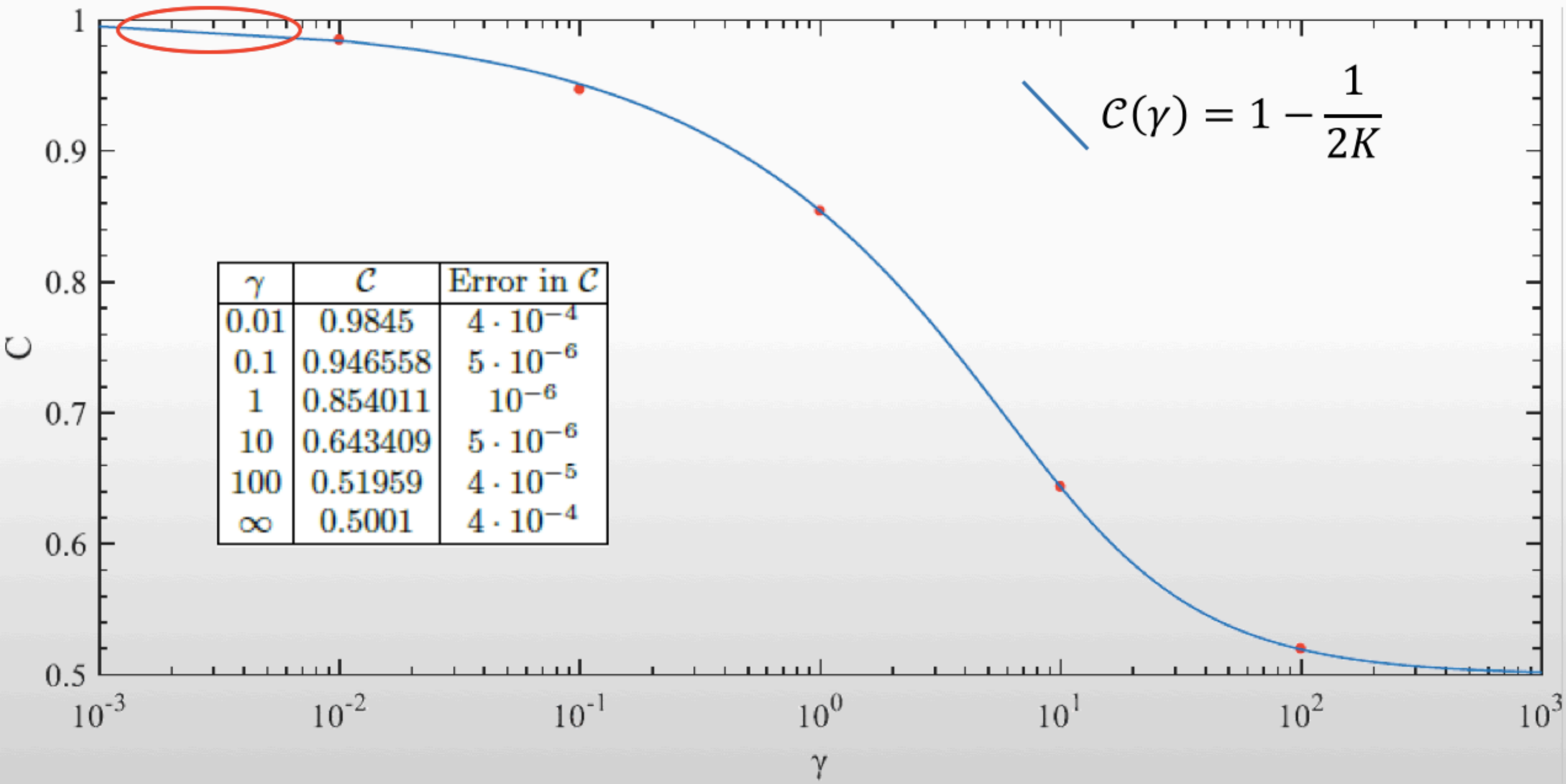
$$\rho_B(z) = \rho_B^{SD}(z)\,\phi^{SD}(z) + \rho_B^{LD}(z)\,\phi^{LD}(z)$$

$$\phi^{SD}(z) = \left[1 - tgh\left(\frac{z}{\alpha}\right)\right]\left[1 - tgh\left(\frac{z^{3/2}}{\beta}\right)\right]$$

$$\phi^{LD}(z) = tgh\left(\frac{z}{\eta}\right) tgh\left(\frac{z}{\omega}\right)$$



Lieb-Liniger Bosons



γ	$\mathcal{C}$	Error in $\mathcal{C}$
0.01	0.9845	$4 \cdot 10^{-4}$
0.1	0.946558	$5 \cdot 10^{-6}$
1	0.854011	10^{-6}
10	0.643409	$5 \cdot 10^{-6}$
100	0.51959	$4 \cdot 10^{-5}$
∞	0.5001	$4 \cdot 10^{-4}$

$$\mathcal{C}(\gamma) = 1 - \frac{1}{2K}$$

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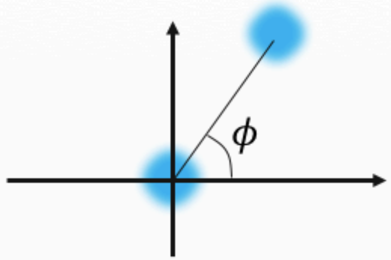
Lieb-Liniger Model with anyons



$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} + 2c \sum_{1 \le i < j \le N} \delta\left(z_i - z_j\right)$$

$$\chi_N^\kappa(z_1, \dots, z_i, z_{i+1}, \dots, z_N) = e^{-i\pi\kappa\, sign(z_i - z_{i+1})} \chi_N^\kappa(z_1, \dots, z_{i+1}, z_i, \dots, z_N)$$

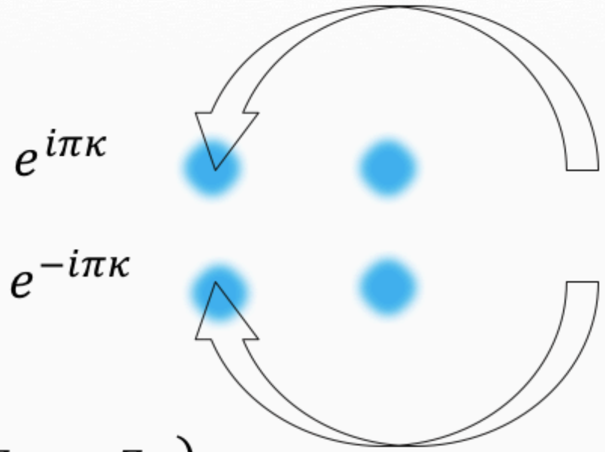
$$\chi_N^\kappa\left(z_1, \dots, z_i, \dots, z_j, \dots, z_N\right) = e^{-i\pi\kappa\left[\sum_{k=i+1}^{j} sign(z_i - z_k) - \sum_{k=i+1}^{j-1} sign(z_j - z_k)\right]} \cdot$$

$$\cdot\, \chi_N^\kappa\left(z_1, \dots, z_j, \dots, z_i, \dots, z_N\right)$$

κ = Statistical parameter

A. Kundu,
Phys. Rev. Lett., 83 (1999) 1275

Lieb-Liniger Model with anyons



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$$\cdot\, \chi_N^\kappa\left(z_1, \dots, z_j, \dots, z_i, \dots, z_N\right)$$

κ = Statistical parameter $\begin{cases} 0 & \text{Bosons} \\ 1 & \text{Fermions,} \end{cases}$

A. Kundu,
Phys. Rev. Lett., 83 (1999) 1275

Lieb-Liniger Model with anyons

Twisted BC: $\chi_N^\kappa(0, x_2, \dots) = e^{i\pi\kappa(N-1)}\chi_N^\kappa(L, x_2, \dots)$ $\longleftrightarrow$ Periodic BC: $\chi_N^0(0, x_2, \dots) = \chi_N^0(L, x_2, \dots)$

O. I. Pâţu, V. E. Korepin, D. V. Averin, *J. Phys. A, 40 (2007) 14963*

$$\lambda_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right) + \frac{2}{L}\sum_{k=1}^{N} \arctan\left(\frac{\lambda_j - \lambda_k}{c'}\right), \qquad j = 1, \dots, N$$

$$c' = \frac{c}{\cos(\pi\kappa/2)} > 0$$

$$\chi_N^\kappa(z_1, \dots, z_N) = \mathcal{N} \exp\left(i\frac{\pi\kappa}{2}\sum_{j<k} sign(z_j - z_k)\right)\det\left(e^{i\lambda_j z_m}\right) \cdot$$

$$\cdot \prod_{n<l}[\lambda_l - \lambda_n - ic' sign(z_l - z_n)]$$

M. T. Batchelor, X.-W. Guan, N. Oelkers, *Phys. Rev. Lett., 96 (2006) 210402*
M. T. Batchelor, X.-W. Guan, J.-S. He, *J. Stat. Mech., (2007) P03007*

Off-Diagonal Long-Range Order

$$\rho_A(z,z') = N \int_0^L dz_2 \dots dz_N \, [\chi_N^\kappa(z, z_2, \dots, z_N)]^* \, \chi_N^\kappa(z', z_2, \dots, z_N)$$

$$\chi_N^\kappa(z+L, z_2, \dots, z_N) = e^{-i\pi\kappa(N-1)} \chi_N^\kappa(z, z_2, \dots, z_N)$$

$$\rho_A(z+L) = e^{i\pi(1-\kappa)(N-1)} \rho_A(z)$$

Homogeneous System
&
Periodicity in OBDM

$\Rightarrow$ $\kappa = \frac{m}{n} \in \mathbb{Q}$

$n, m = 0,1,2, \dots, m \leq n$

$$\lambda_0 \propto N^{C(\kappa)} \propto \frac{n_A(p = -p_F \kappa)}{L}$$

Predictions for $\mathcal{C}(\kappa)$

$$n_A(p=0) \propto N^{\mathcal{C}(\kappa)}$$

$$n_A(p) = \frac{L}{2\pi}\int_0^L \rho_A(z)\, e^{ipz}\, dz$$

Large distance $\rho_A(z \gg 1)$ $\Longrightarrow$ "Small" momenta $n_A(p \approx -p_F\kappa)$

$\boxed{\kappa = 0}$ $\quad \mathcal{C}(\kappa = 0, \gamma) = 1 - \frac{1}{2K}$ $\quad K(\gamma) = \frac{v_F}{s(\gamma)}$

$\boxed{\kappa \text{ generic}}$ $\quad \frac{n_A(p)}{L} \propto \frac{1}{(p + p_F\kappa)^{1 - \frac{1}{2K} - \frac{K\kappa^2}{2}}} \Longrightarrow \boxed{\mathcal{C}(\kappa, \gamma) = 1 - \frac{1}{2K} - \frac{K\kappa^2}{2}}$

P. Calabrese, M. Mintchev, *Phys. Rev. B, 75 (2007) 233104*

Hard-Core Anyons $\gamma \to \infty$

$$\chi_N^{\kappa}(z_1, \dots, z_N) = \left[\prod_{1 \le i < j \le N} A(z_j - z_i)\right] \chi_N^1(z_1, \dots, z_N) \qquad \kappa = \frac{m}{n} \in \mathbb{Q}$$

Anyon – Fermi mapping: $A(z_j - z_i) = \left[\theta(z_j - z_i) + \theta(z_i - z_j) e^{i\pi(1-\kappa)}\right]$ $\qquad \theta(0) = 0$

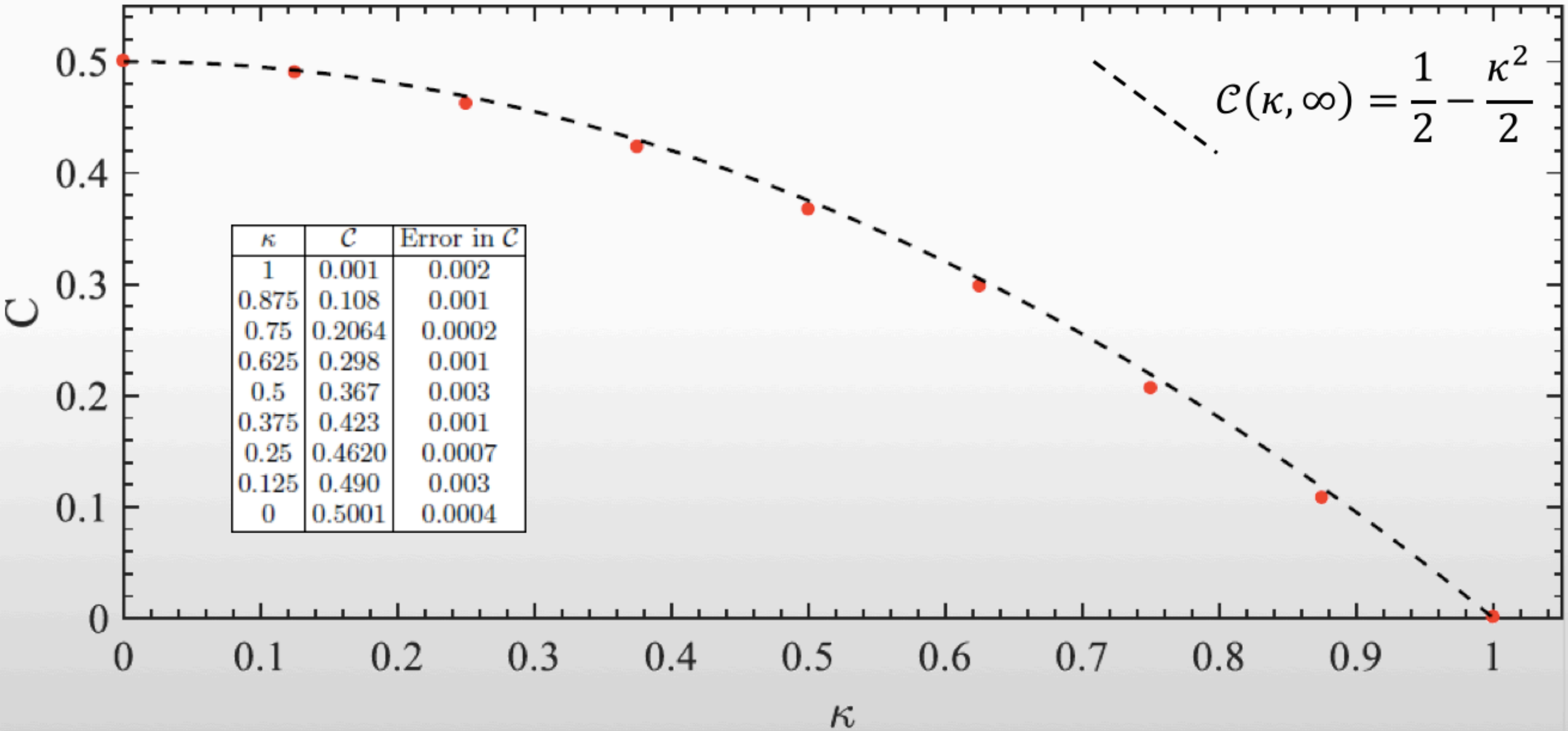
$\boxed{\kappa = 0}$ Boson – Fermi mapping: $A(z_j - z_i) = sign(z_j - z_i)$

$$\rho_A(t) = \det\left[\frac{2}{\pi}\int_0^{2\pi} d\tau\, e^{i(j-l)\tau} A(\tau - t) \sin\left(\frac{\tau - t}{2}\right) \sin\left(\frac{\tau}{2}\right)\right]_{j,l=1,\dots,N-1} \qquad t \equiv \frac{2\pi x}{L}$$

M.D. Girardeau, *Phys. Rev. Lett., 97 (2006) 210401*

R. Santachiara, P. Calabrese, *J. Stat. Mech., (2008) P06005*

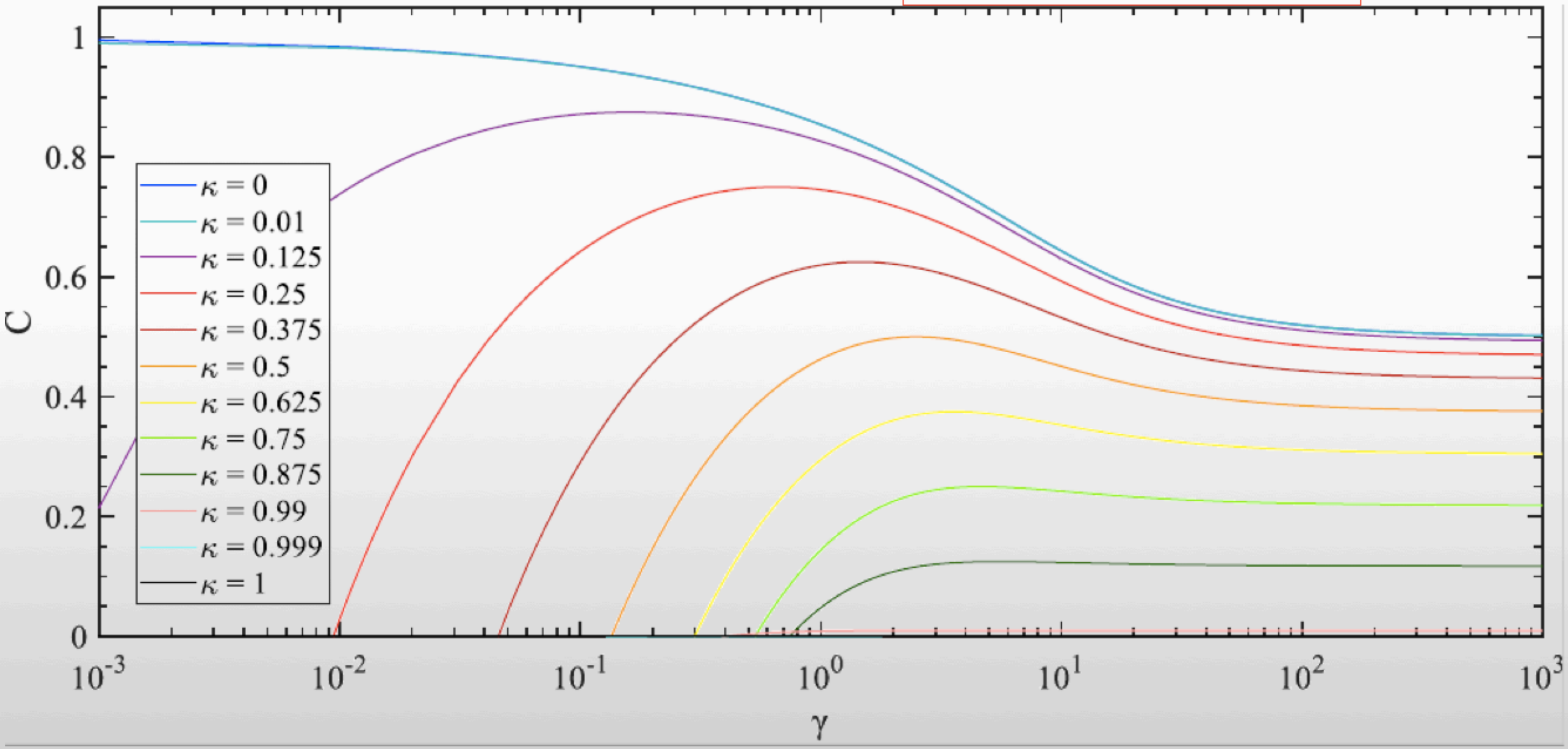
Hard-Core Anyons $\mathcal{C}(\kappa, \gamma \to \infty)$



κ	$\mathcal{C}$	Error in $\mathcal{C}$
1	0.001	0.002
0.875	0.108	0.001
0.75	0.2064	0.0002
0.625	0.298	0.001
0.5	0.367	0.003
0.375	0.423	0.001
0.25	0.4620	0.0007
0.125	0.490	0.003
0	0.5001	0.0004

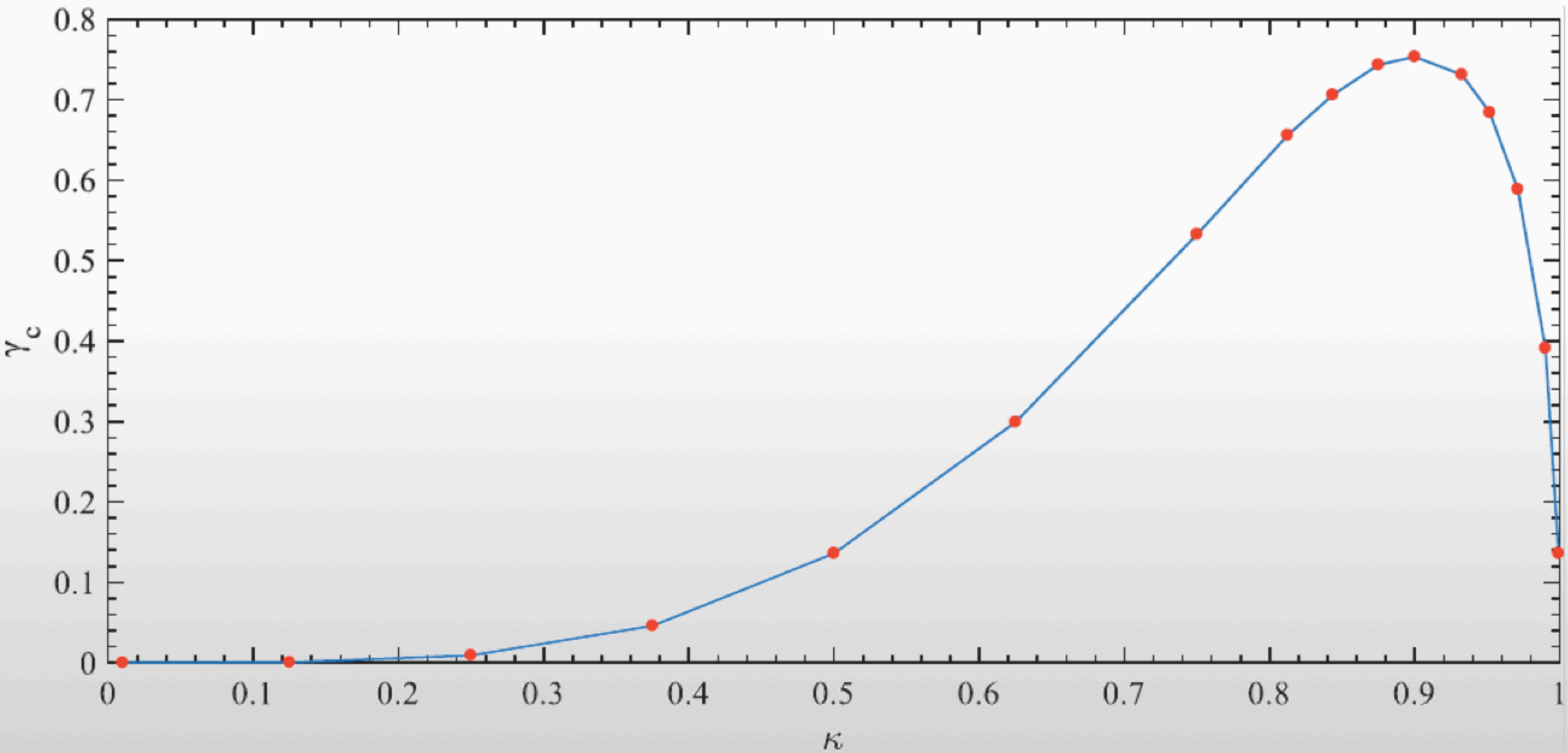
Predictions for $\mathcal{C}(\kappa, \gamma)$

$$\mathcal{C}(\kappa, \gamma) = 1 - \frac{1}{2K} - \frac{K\kappa^2}{2}$$



Critical Coupling

$$\mathcal{C}(\kappa, \gamma_c) = 0$$



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Conclusions and Outlook

- ODLRO in terms of occupation numbers
- Quantify deviations from ODLRO in 1D Systems
- Bosonization and Harmonic Fluid Approach as check
- Inhomogeneous LL bosons (finite coupling), Finite Temperature, Experimental Results …