# Non-linear Schrödinger equation with point interactions

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Based on joint works with Vladimir Georgiev, Alessandro Michelangeli and Alessandro Olgiati.

# Heuristic model

Formally, consider the equation

$$i\partial_t u = -\Delta_x + \sum_{j=1}^{N} \mu_j \, \delta(x - y_j) + \mathcal{N}(u),$$

where $\{y_1, \ldots y_N\}$ are distinct points in $\mathbb{R}^d$, which supports delta-like interactions of strenght $\{\mu_1, \ldots \mu_N\}$, and $\mathcal{N}(u)$ is a non-linear interaction potential.

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**Why we consider point interactions?**

**Why we consider non-linear potentials?**

**How to give a rigorous meaning to the equation?**

# Point interactions, motivations

$$-\Delta_x + \sum_{j=1}^{N} \mu_j \, \delta(x - y_j)$$

provides an heuristic model for a quantum particle subject to a "contact potential", created by point sources of strenght $\mu_y$ centered at $y_j$.

- Kronig and Penney (1931) consider the 1D case as a model for a non-relativistic electron moving in a fixed crystal lattice.
- Bethe, Peierls (1935) and Thomas (1935) consider the 3D case with $y = 0$. Introducing the center of mass and relative coordinates, this provides a model for a deuteron with idealized zero-range nuclear force between the nucleons.
- Appears in many contexts: nuclear physics, solid state physics etc.
- Provide "solvable" approximation of more complicated and realistic phenomena, governed by very short range interactions

# Non-linear potentials, motivations

Consider the 3D *many-body* Hamiltonian

$$H_N := \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{1\leq j<k\leq N} w_N(x_j - x_k),$$

where $w_N$ is a pontential governing the interaction between the particles. Assume *Bose-Einstein condensation* at time $t = 0$:

$$\psi_0^{(N)}(x_1, \dots, x_N) \approx \prod_{j=1}^{N} u_0(x_j), \quad N \gg 1.$$

Then we have condensation at any time $t > 0$:

$$\left(e^{itH_N}\psi_0^{(N)}\right)(x_1, \dots, x_N) \approx \prod_{j=1}^{N} u(t, x_j), \quad N \gg 1,$$

$$\begin{cases} i\partial_t u = -\Delta u + \mathcal{N}(u) \\ u(0,x) = u_0(x), \end{cases} \qquad \mathcal{N}(u) = \begin{cases} (w * |u|^2)u & w_N = N^{-1}w \\ |u|^2 u & w_N = N^2 w(Nx) \end{cases}$$

# Non-linear potential with point interactions

$$H_{N,\varepsilon} := \sum_{j=1}^{N} \big( -\Delta_{x_j} + V_\varepsilon(x_j) \big) + \sum_{1 \le j < k \le N} w_N(x_j - x_k),$$

where $V_\varepsilon$ are smooth potentials, shrinking around the origin in such a way to create a delta-like profile as $\varepsilon \to 0$. Assume Bose-Einstein condensation at time $t = 0$.

- **Is condensation preserved, at least for short times?**
- **Can we rigorous prove that the evolution of the condensate is governed by the equation**

$$i\partial_t u = -\Delta u + \mu\delta(x) + \mathcal{N}(u)$$

**in the limit** $N \to +\infty$ **and** $\varepsilon \to 0$?

Work in progress with A. Michelangeli and A. Olgiati. As a first step, we need to show existence of solutions for the limit equation.

Assume, for simplicity, a single center at the origin.

- In dimension one, consider the quadratic form

$$Q(f,g) := \int_{\mathbb{R}} \overline{\partial_x f} \cdot \partial_x g \, dx + \mu \overline{f(0)} g(0), \quad f,g \in H^1(\mathbb{R}).$$

From $Q$, we realise $-\Delta_x + \mu\delta(x)$ as a *self-adjoint operator* on $L^2(\mathbb{R})$.

- In higher dimension, we need a different approach. Suppose $d = 3$. The *symmetric* operator $-\Delta|_{C_0^\infty(\mathbb{R}^3\setminus\{0\})}$ has a one-parameter family of *self-adjoint extension* $\{-\Delta_\alpha\}_{\alpha\in(-\infty,+\infty]}$. For $\lambda > 0$,

$$\mathcal{D}(-\Delta_\alpha) = \left\{ \psi \in L^2(\mathbb{R}^3) \, : \, \psi = F_\lambda + \frac{F_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}} \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|} \, : \, F_\lambda \in H^2(\mathbb{R}^3) \right\}$$

$$(-\Delta_\alpha + \lambda)\psi = (-\Delta + \lambda)F_\lambda$$

# Asymptotics of the wave function

- The parameter $\alpha$ is related to the *scattering lenght* of the system at the centre of interaction. Indeed, a generic element $\psi \in \mathcal{D}(-\Delta_\alpha)$ satisfies the so called *Bethe-Peierls contact condition*

$$\psi(x) \underset{x\to 0}{\sim} \frac{1}{|x|} - \frac{1}{s}, \quad s = -(4\pi\alpha)^{-1},$$

  which is typical for the low-energy behaviour of an eigenstate of the Schrödinger equation for a quantum particle subject to a very short (virtually zero) range potentials, centered at the origin and with s-wave scattering lenght $s$.
- When $\alpha = +\infty$, then no actual interaction is present at the origin (the s-wave scattering lenght is zero), and we recover the free Laplacian in $L^2(\mathbb{R}^3)$.

# Approximation with regular operators

Let $V$ smooth and compactly supported, and assume that $-\Delta + V$ has a *zero energy resonance*, namely a funtion $\psi \in L^1 \setminus L^2$ such that

$$(-\Delta + V)\psi = 0.$$

Existence of a zero energy resonance is related to *confining* property of the potential $V$. In particular, $V$ must have a negative part.
Define, for $\varepsilon > 0$ and a function $\lambda : \mathbb{R} \to \mathbb{R}$, with $\lambda(0) = 1$,

$$V_\varepsilon := \frac{\lambda(\varepsilon)}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right)$$

The potential is shrinking around the origin. N.B. the scaling is **not** that of a delta function, but is *weaker*. We have, in a suitable resolvent sense

$$\lim_{\varepsilon \to 0} (-\Delta + V_\varepsilon) = -\Delta_\alpha$$

# Rigorous formulation of the equation

Consider the Cauchy problem

$$\begin{cases} i\partial_t u = -\Delta_\alpha u + \mathcal{N}(u), & t \in \mathbb{R}, x \in \mathbb{R}^3, \\ u(0,\cdot) = f \in X \end{cases}$$

where $X$ is a suitable Hilbert space, for example $L^2(\mathbb{R}^3)$.

- Since $-\Delta_\alpha$ is self-adjoint, we have a unitary evolution $e^{-it\Delta_\alpha} f$.
- Integral formulation of the equation:

$$u(t,x) = e^{it\Delta_\alpha} f - i \int_0^t e^{i(t-s)\Delta_\alpha} \mathcal{N}(u)(s) ds$$

- We search for solution $u \in \mathcal{C}(I, X)$ of the integral equation, for some time interval $I$.

# Energy space

Define, for $\alpha > 0$, the Banach space

$$H^1_\alpha := \mathcal{D}((-\Delta_\alpha)^{1/2}) \quad \|\psi\|_{H^1_\alpha} := \|(-\Delta_\alpha + 1)^{1/2}\psi\|_{L^2(\mathbb{R}^3)}$$

- When $\alpha = +\infty$, we recover the Sobolev space $H^1(\mathbb{R}^3)$.
- We have an explicit characterisation ([Georgiev, Michelangeli, S.] for a discussion of the general fractional case)

$$H^1_\alpha = \left\{ \psi \in L^2(\mathbb{R}^3) \, : \, \psi = F_\lambda + c\frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|} \, : \, F_\lambda \in H^1(\mathbb{R}^3), c \in \mathbb{C} \right\}$$

$$\left\| F_\lambda + c\frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|} \right\|^2_{H^1_\alpha} \sim \|F\|^2_{H^1} + (1+\alpha)|c|^2$$

- The $\|\cdot\|_{H^1_\alpha}$-norm is preserved by the unitary evolution $e^{-it\Delta_\alpha}$.
- We will use $H^1_\alpha$ as the energy space also for the non-linear problem.

# Existence of solutions

**Theorem (Michelangeli, Olgiati, S., 2018)**

Let $w = |x|^{-\gamma}$, with $0 < \gamma < \frac{1}{2}$. Then the Cauchy problem

$$\begin{cases} i\partial_t u = -\Delta_\alpha u + (w * |u|^2)u, & t \in \mathbb{R}, x \in \mathbb{R}^3, \\ u(0,\cdot) = f \in H^1_\alpha \end{cases}$$

has a unique solution $u \in \mathcal{C}([0,T^*), H^1_\alpha)$, defined on a maximal time interval $[0, T^*)$.

We have the *blow-up* alternative:

$$T^* < +\infty \iff \lim_{t \uparrow T^*} \|u(t)\|_{H^1_\alpha} = +\infty$$

As long as $\|u(t)\|_{H^1_\alpha}$ stay bounded, the solution can be extended in time.

# Global solutions

Defone the *mass* and the *energy*:

$$\mathcal{M}(u) := \int_{\mathbb{R}^3} |u|^2 dx$$

$$\mathcal{E}(u) :=< -\Delta_\alpha u, u > + \frac{1}{2} \int_{\mathbb{R}^3} (w * |u|^2)|u|^2 dx$$

- Formally, if $u$ is a solution of the Cauchy problem, then $\mathcal{M}(u(t))$ $\mathcal{E}(u(t))$ are *conserved*.
- To rigorous justify energy conservation, we need the additional asusmption that $w$ and $u$ are *spherically symmetric* (only a mathematical issue or there is a physical meaning?)
- We want to use mass and energy conservation to find global in time solution (condensation is preserved also for large times).

Assume that $w \geqslant 0$ (repulsive interaction). Then

$$\|u(t)\|^2_{H^1_\alpha} \approx \mathcal{M}(u(t)) + < -\Delta_\alpha u, u > \\ \leq \mathcal{M}(u(t)) + \mathcal{E}(u(t)) = \mathcal{M}(f) + \mathcal{E}(f)$$

Hence $\|u(t)\|_{H^1_\alpha}$ remains bounded, and by the blow-up alternatives the solution is global.

- Also if $\widehat{w} \geqslant 0$ (physical meaning of this condition?) the potential energy is positive, whence global in time solution.
- In general, for attractive $w$, the dynamic is more complicated: blow-up solutions, bound states.
- It would be interesting to investigate the nature of these solutions, and how they depends on the presence of a point interaction.

Thank you for your attention